

A-Level Maths

Pure P23

Numerical Methods

Learning and Revision Booklet

Suitable for students studying Pearson Edexcel A-Level Mathematics

Original explanations, practice questions and worked answers.

Method Marks

Free learning resource

About this booklet

Important note

This resource is independently created. It is not affiliated with, endorsed by, or produced by Pearson Edexcel. All questions, examples, diagrams and worked solutions are original.

This booklet teaches the numerical methods used in A-Level Pure Mathematics. These methods are used when an exact algebraic solution is difficult or impossible, so you approximate the answer carefully and check whether the result is sensible.

How to use it

Step	What to do
1	Read the key facts before trying the examples.
2	Cover the working and try each example yourself.
3	Complete the guided practice before moving to harder questions.
4	Mark your work using the fully worked answers.
5	Go back to any section where your rounding or iteration setup was unclear.

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Section	Topic
1	Sign change and locating roots
2	Fixed point iteration
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Before you start

You should already be confident with substituting into functions, solving equations, using a calculator accurately, differentiating basic functions and interpreting graphs.

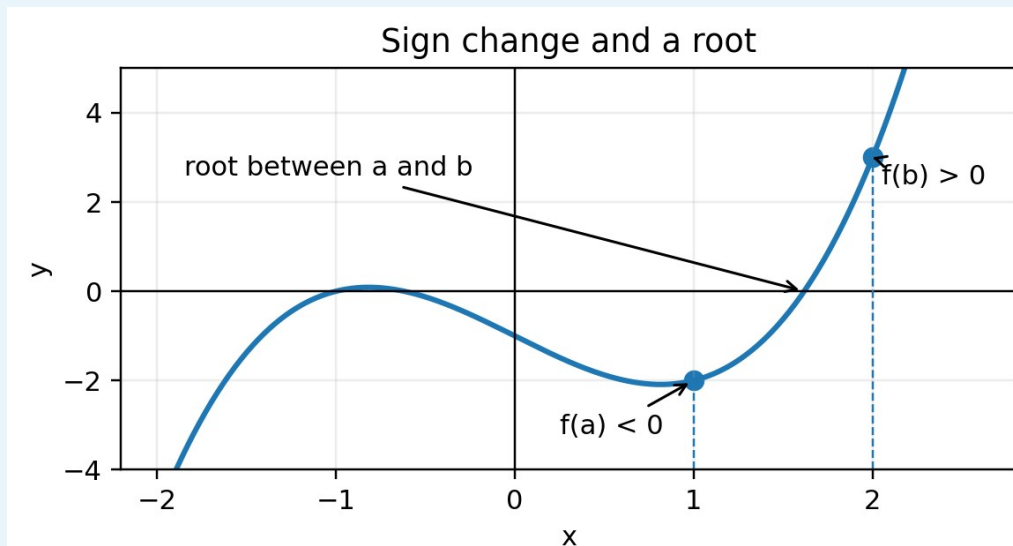
1. Sign change and locating roots

A root of $f(x) = 0$ is a value of x where the graph crosses or touches the x -axis. If a continuous function changes sign between two x -values, there is at least one root between them.

Key fact

For a continuous function:

$$f(a)f(b) < 0 \Rightarrow \text{a root lies between } a \text{ and } b$$

Diagram: sign change**Method: showing a root lies in an interval**

Step	What to do
1	Substitute the lower value into the function.
2	Substitute the upper value into the function.
3	Check whether the signs are different.
4	State that the function is continuous.
5	Conclude that a root lies in the interval.

Example 1: showing a root lies between two values

Question: Show that a root lies in the interval.

Working:

Show that $f(x) = x^3 - 2x - 1$ has a root between 1 and 2.

$$f(1) = 1^3 - 2(1) - 1 = -2$$

$$f(2) = 2^3 - 2(2) - 1 = 3$$

$f(1) < 0$ and $f(2) > 0$, so there is a sign change.

Guided practice 1

Question	Space for working
1. Show that $x^3 - x - 1 = 0$ has a root between 1 and 2.	
2. Show that $x^2 - 3 = 0$ has a root between 1 and 2.	
3. Show that $e^x - 4 = 0$ has a root between 1 and 2.	
4. Show that $x^3 + x - 5 = 0$ has a root between 1 and 2.	
5. Explain why the function must be continuous for this argument to work.	
6. Find an interval of width 1 that contains a root of $x^3 - 6x + 2 = 0$.	

2. Fixed point iteration

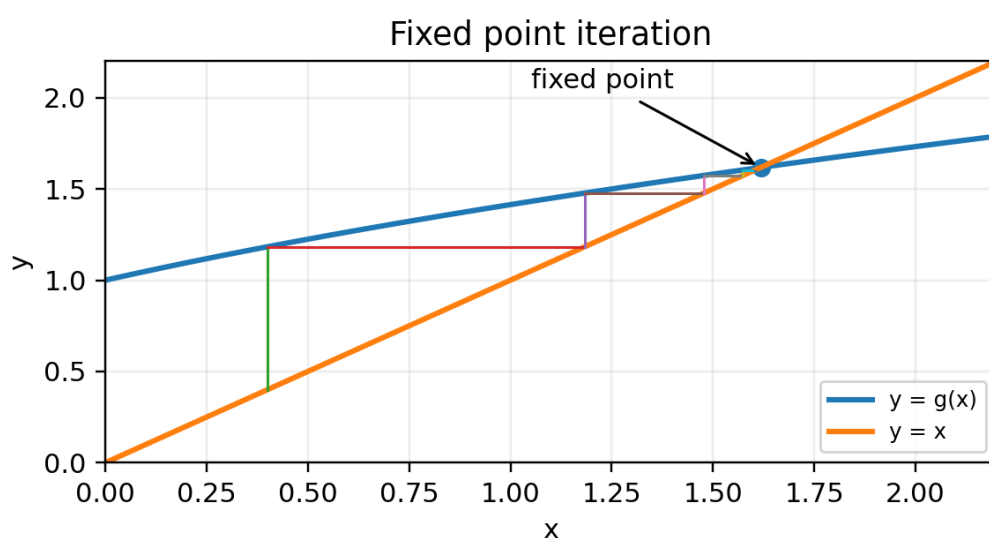
Fixed point iteration rewrites an equation in the form $x = g(x)$, then repeatedly uses the formula to improve an approximation.

Key fact

A fixed point iteration uses:

$$x_{n+1} = g(x_n)$$

Diagram: fixed point iteration



Method: fixed point iteration

Step	What to do
1	Rearrange the equation into the form $x = g(x)$.
2	Choose or use the starting value x_0 .
3	Substitute into the iteration formula to find x_1 .
4	Repeat to find x_2 , x_3 and so on.
5	Round only at the end unless the question tells you otherwise.

Example 2: fixed point iteration

Question: Use the iteration formula for three iterations.

Working:

Use $x_{n+1} = \sqrt{x_n + 1}$ with $x_0 = 1$ for three iterations.

$$x_1 = \sqrt{1 + 1} = 1.414213\dots$$

$$x_2 = \sqrt{1.414213\dots + 1} = 1.553773\dots$$

$$x_3 = \sqrt{1.553773\dots + 1} = 1.598053\dots$$

So after three iterations, $x \approx 1.598$ to 3 d.p.

Guided practice 2

Question	Space for working
1. Use $x_{n+1} = \sqrt{2 + x_n}$, $x_0 = 1$, for three iterations.	
2. Use $x_{n+1} = 1 + 1/x_n$, $x_0 = 2$, for four iterations.	
3. Use $x_{n+1} = (x_n + 2/x_n)/2$, $x_0 = 1.5$, for three iterations.	
4. Rearrange $x^2 - x - 1 = 0$ into a suitable iteration formula.	
5. Explain what it means for an iteration to converge.	
6. Explain why different rearrangements may behave differently.	

3. Newton-Raphson method

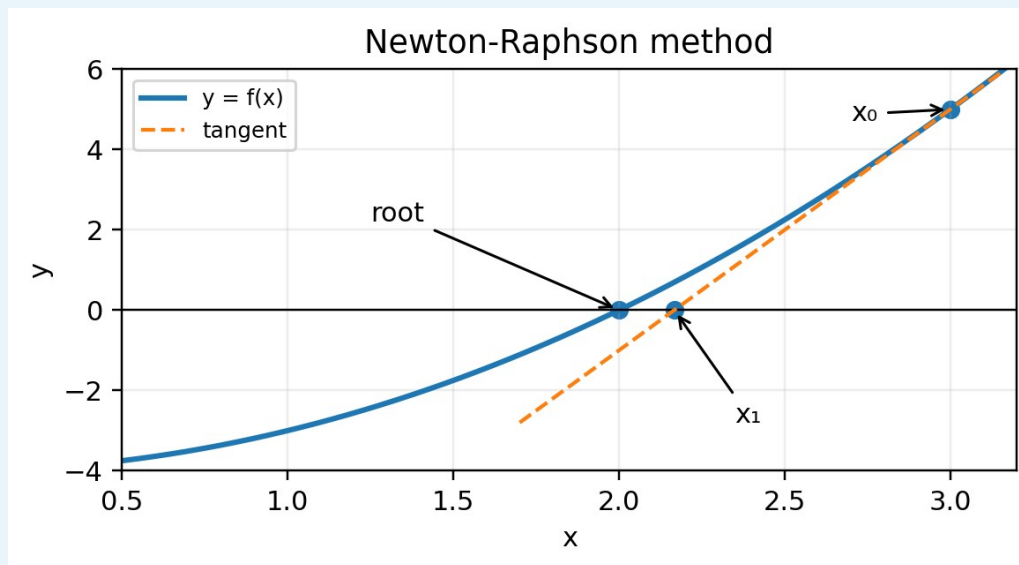
Newton-Raphson uses tangents to improve an approximation to a root. It is usually fast when the starting value is sensible and the function behaves well near the root.

Key fact

Newton-Raphson uses:

$$X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)}$$

Diagram: Newton-Raphson



Method: Newton-Raphson

Step	What to do
1	Define $f(x)$ and find f' prime of x .
2	Choose or use the starting value.
3	Substitute into the Newton-Raphson formula.
4	Repeat until the required accuracy is reached.
5	Check your answer by substituting into the original equation.

Example 3: Newton-Raphson

Question: Use Newton-Raphson to approximate the root.

Working:

Use Newton-Raphson to solve $x^2 - 5 = 0$ starting with $x_0 = 2$.

$$f(x) = x^2 - 5, \quad f'(x) = 2x$$

$$x_1 = 2 - \frac{-1}{4} = 2.25$$

$$x_2 = 2.25 - \frac{0.0625}{4.5} = 2.236111\dots$$

So $\sqrt{5} \approx 2.236$ to 3 d.p.

Guided practice 3

Question	Space for working
1. Use Newton-Raphson to solve $x^2 - 2 = 0$, starting with $x_0 = 1.5$.	
2. Use Newton-Raphson to solve $x^3 - 5 = 0$, starting with $x_0 = 2$.	
3. Use Newton-Raphson to solve $\cos x - x = 0$, starting with $x_0 = 0.7$.	
4. Find f' of x for $f(x) = x^3 + 2x - 7$.	
5. Explain why a poor starting value can cause problems.	
6. State one reason Newton-Raphson may fail or give an unsuitable approximation.	

4. The trapezium rule

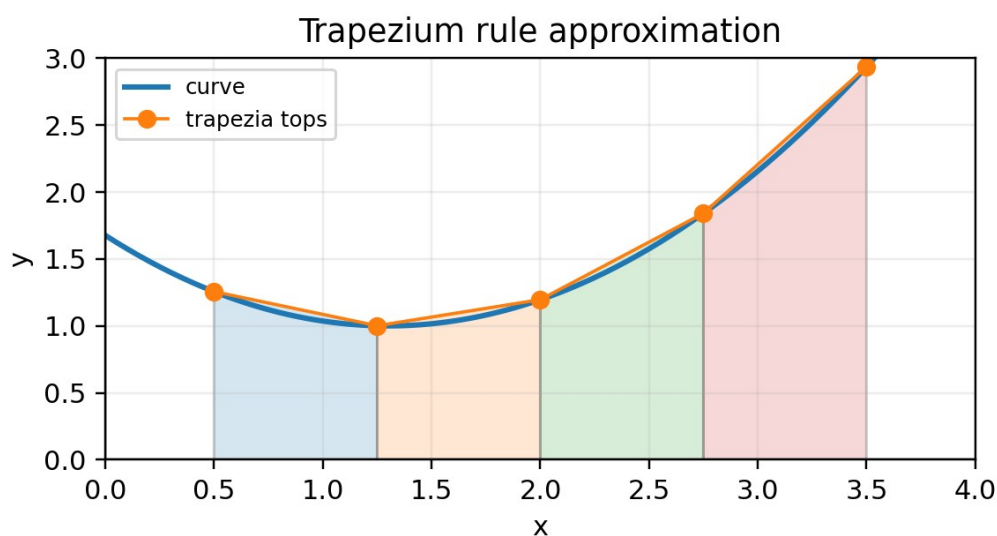
The trapezium rule estimates the area under a curve by splitting the region into trapezia. It is useful when exact integration is difficult or when values are given in a table.

Key fact

For n equal strips of width h :

$$\int_a^b y \, dx \approx \frac{h}{2} [y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1})]$$

Diagram: trapezium rule



Method: trapezium rule

Step	What to do
1	Check that the x-values are equally spaced.
2	Find the strip width h .
3	List the y-values in order.
4	Use the trapezium rule formula carefully.
5	Round to the accuracy requested in the question.

Example 4: trapezium rule

Question: Estimate the integral using the trapezium rule.

Working:

Use the trapezium rule with $h = 1$ to estimate $\int_0^4 (x^2 + 1) dx$.

$$y_0 = 1, y_1 = 2, y_2 = 5, y_3 = 10, y_4 = 17$$

$$\begin{aligned} \text{Estimate} &= \frac{1}{2}[1 + 17 + 2(2 + 5 + 10)] \\ &= \frac{1}{2}(52) = 26 \end{aligned}$$

Guided practice 4

Question	Space for working
1. Use the trapezium rule with $h = 1$ to estimate the area under $y = x^2$ from $x = 0$ to $x = 4$.	
2. Use the trapezium rule with $h = 0.5$ for $y = x^3$ from $x = 0$ to $x = 2$.	
3. A table gives y-values 2, 3, 5, 8, 12 at equally spaced x-values with $h = 1$. Estimate the area.	
4. Use five ordinates to estimate the area under $y = 1/x$ from $x = 1$ to $x = 3$.	
5. Explain when the trapezium rule is likely to overestimate an area.	
6. Explain why using more strips usually improves the estimate.	

5. Accuracy, rounding and stopping rules

Numerical methods produce approximations, so you need to think carefully about rounding, decimal places, significant figures and when an iterative process should stop.

Important

Do not round too early. Keep calculator values during the working and round the final answer to the required accuracy.

Method: deciding when to stop

Step	What to do
1	Check the accuracy requested in the question.
2	Continue the method until consecutive approximations agree to that accuracy.
3	Keep enough decimal places in the working.
4	Substitute the result into the original equation as a check.
5	Write the final answer clearly with the correct units or accuracy.

Example 5: checking convergence

Question: Use the iteration to approximate the value.

Working:

The iteration $x_{n+1} = \frac{1}{2}(x_n + \frac{3}{x_n})$ is used to find $\sqrt{3}$.

Starting with $x_0 = 2$, $x_1 = \frac{1}{2}(2 + \frac{3}{2}) = 1.75$

$x_2 = \frac{1}{2}(1.75 + \frac{3}{1.75}) = 1.732142\dots$

The values are getting closer to 1.732...

Guided practice 5

Question	Space for working
1. Explain why 1.732142 should be rounded to 1.73 to 3 significant figures.	
2. Two consecutive approximations are 2.45641 and 2.45648. Give the value to 3 decimal places.	
3. Explain why calculator memory is useful during numerical methods.	
4. State a stopping rule for giving an answer to 4 decimal places.	
5. A root is approximately 0.739085. Give it to 3 significant figures and 4 decimal places.	
6. Explain why an answer from a numerical method should be checked in the original equation.	

6. Mixed exam-style questions

Question	Space for working
1. [3 marks] Show that $x^3 - 4x + 1 = 0$ has a root between 0 and 1.	
2. [4 marks] Use $x_{n+1} = \sqrt{3 - x_n}$, starting with $x_0 = 1$, for four iterations.	
3. [5 marks] Use Newton-Raphson to solve $x^2 - 7 = 0$, starting with $x_0 = 3$. Give your answer to 3 d.p.	
4. [4 marks] Use the trapezium rule with $h = 1$ to estimate	

Question	Space for working
the integral of $y = x^2 + 2$ from $x = 0$ to $x = 4$.	
5. [3 marks] Explain why the trapezium rule is an approximation rather than an exact method in most cases.	
6. [4 marks] Rearrange $x^2 - 3x - 1 = 0$ into an iteration formula and use $x_0 = 3$ for three iterations.	
7. [5 marks] Use Newton-Raphson once for $f(x) = x^3 - 2x - 5$ starting with $x_0 = 2$.	
8. [4 marks] The values of y are 1, 1.8, 3.2, 5.1, 8.0 at equally spaced x -values with $h = 0.5$. Use the trapezium rule.	
9. [3 marks] Explain what it means for an iteration to converge.	
10. [4 marks] Use a sign change to show that $e^x - 3x$ has a root between 0 and 1.	

7. Challenge questions

Question	Space for working
C1. Use two iterations of Newton-Raphson to solve $x^3 - x - 1 = 0$ starting with $x_0 = 1.3$.	
C2. The iteration $x_{n+1} = 2/x_n$ is started with $x_0 = 1$. Explain what happens and why this is not convergence.	
C3. A trapezium rule estimate uses 6 ordinates over an interval from $x = 1$ to $x = 3$. Find h and write the full estimate formula.	
C4. Rearrange $x = \cos x$ into a fixed point iteration and use $x_0 = 0.7$ for three iterations.	
C5. Use Newton-Raphson to estimate the positive root of $x^3 - 9 = 0$ starting with $x_0 = 2$.	

8. Common mistakes

Mistake	How to fix it
Claiming a root from one substitution only.	You need two values with opposite signs and a continuous function.
Rounding after every iteration.	Keep the full calculator value until the final answer.
Using Newton-Raphson without differentiating correctly.	Find f prime of x carefully before substituting into the formula.
Forgetting the first and last y -values in the trapezium rule.	Use y_0 and y_n once, with the middle ordinates doubled.
Using unequal x -values in the trapezium rule formula.	The standard formula needs equal spacing. Check h first.
Giving too many or too few decimal places.	Read the required accuracy and state the answer clearly.

9. Fully worked answers

Guided practice 1

1. $f(1) = -1$ and $f(2) = 5$, so there is a root between 1 and 2.
2. $f(1) = -2$ and $f(2) = 1$, so there is a root between 1 and 2.
3. $e^1 - 4$ is negative and $e^2 - 4$ is positive, so there is a root between 1 and 2.
4. $f(1) = -3$ and $f(2) = 5$, so there is a root between 1 and 2.
5. Without continuity, the graph might jump over the x-axis rather than crossing it.
6. $f(0) = 2$ and $f(1) = -3$, so one suitable interval is $0 < x < 1$.

Guided practice 2

1. $x_1 = 1.732\dots$, $x_2 = 1.931\dots$, $x_3 = 1.983\dots$
2. $x_1 = 1.5$, $x_2 = 1.666\dots$, $x_3 = 1.6$, $x_4 = 1.625$.
3. $x_1 = 1.416\dots$, $x_2 = 1.414215\dots$, $x_3 = 1.414213\dots$
4. One possible formula is $x_{n+1} = 1 + 1/x_n$.
5. Convergence means the sequence of approximations settles towards a fixed value.
6. Some rearrangements move towards the root; others move away or oscillate.

Guided practice 3

1. $f(x) = x^2 - 2$. Starting at 1.5 gives $x_1 = 1.416\dots$, $x_2 = 1.41421\dots$, so $x \approx 1.414$.
2. $f(x) = x^3 - 5$. Starting at 2 gives $x_1 = 1.75$, $x_2 \approx 1.71088$, so $x \approx 1.710$ to 3 d.p.
3. $f(x) = \cos x - x$ and f' of $x = -\sin x - 1$. Starting at 0.7 gives about 0.739.
4. f' of $x = 3x^2 + 2$.
5. A poor starting value may lead to a different root, divergence or a tangent that gives a bad next estimate.
6. It may fail if f' of x is zero or very small near the estimate.

Guided practice 4

1. y-values 0, 1, 4, 9, 16. Estimate = 22.
2. y-values at 0, 0.5, 1, 1.5, 2 are 0, 0.125, 1, 3.375, 8. Estimate = 4.5.
3. Estimate = $\frac{1}{2}[2 + 12 + 2(3 + 5 + 8)] = 23$.
4. Use x-values 1, 1.5, 2, 2.5, 3 and apply the trapezium formula.
5. If the curve is concave up, the trapezia may lie above the curve, causing an overestimate.
6. More strips usually follow the curve more closely.

Guided practice 5

1. 1.732142 becomes 1.73 to 3 significant figures because the next digit is 2.
2. Both values round to 2.456 to 3 decimal places.
3. Calculator memory avoids rounding errors during repeated calculations.
4. Continue until consecutive approximations agree to 4 decimal places.
5. 0.739085 is 0.739 to 3 significant figures and 0.7391 to 4 decimal places.
6. Substitution checks that the approximation actually makes the original equation nearly true.

Mixed exam-style questions

1. $f(0) = 1$ and $f(1) = -2$, so there is a root between 0 and 1.
2. $x_1 \approx 1.414$, $x_2 \approx 1.260$, $x_3 \approx 1.319$, $x_4 \approx 1.296$.
3. Newton-Raphson gives approximately 2.646 to 3 d.p.
4. y-values are 2, 3, 6, 11, 18. Estimate = 26.
5. It replaces the curved region with straight-sided trapezia, so it is usually not exact.
6. One formula is $x_{n+1} = 3 + 1/x_n$. Starting with 3 gives about 3.333, 3.300, 3.303.

7. $f(2) = -1$ and f' prime of 2 = 10, so $x_1 = 2.1$.
8. Estimate = $0.25[1 + 8.0 + 2(1.8 + 3.2 + 5.1)] = 7.3$.
9. It means the approximations get closer to a fixed value.
10. $f(0) = 1$ and $f(1) = e - 3$ is negative, so there is a root between 0 and 1.

Challenge questions

- C1. Starting with 1.3 gives $x_1 \approx 1.32564$ and $x_2 \approx 1.32472$.
- C2. The values alternate between 1 and 2, so they do not settle to a single value.
- C3. There are 5 strips, so $h = (3 - 1)/5 = 0.4$. Estimate = $0.2[y_0 + y_5 + 2(y_1 + y_2 + y_3 + y_4)]$.
- C4. Use $x_{n+1} = \cos x_n$. Starting at 0.7 gives about 0.7648, 0.7215, 0.7508.
- C5. Starting at 2 gives $x_1 \approx 2.08333$ and $x_2 \approx 2.08008$, so the positive root is about 2.080.

10. Revision checklist

Skill	Tick
I can use a sign change to locate a root in an interval.	☐
I can explain why continuity matters when locating roots.	☐
I can use fixed point iteration correctly.	☐
I can recognise convergence and possible divergence.	☐
I can use the Newton-Raphson formula.	☐
I can apply the trapezium rule with equal spacing.	☐
I can round numerical method answers to the required accuracy.	☐
I can check an approximate root in the original equation.	☐