

A-Level Maths

Pure P21

Integration Techniques

Learning and Revision Booklet

Suitable for students studying Pearson Edexcel A-Level Mathematics

Original explanations, practice questions and worked answers.

Method Marks

Free learning resource

About this booklet

Important note

This resource is independently created. It is not affiliated with, endorsed by, or produced by Pearson Edexcel. All questions, examples, diagrams and worked solutions are original.

This booklet develops the integration techniques used in A-Level Pure Mathematics. It builds on basic integration and covers reverse chain rule, substitution, integration by parts, partial fractions and common trigonometric integrals.

How to use it

Step	What to do
1	Read the key facts before trying the examples.
2	Cover the working and try each example yourself.
3	Complete the guided practice before moving to harder questions.
4	Mark your work using the fully worked answers.
5	Go back to any section where your working was unclear or incomplete.

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Before you start

You should already be confident with basic integration, differentiating powers, differentiating exponentials and trigonometric functions, and simplifying algebraic fractions.

1. Reverse chain rule

The reverse chain rule is used when the integrand contains a function and its derivative. You are looking for the inside function and a multiple of its derivative.

Key facts

These forms are useful when the derivative of the inside function is already present.

$$\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

Method: reverse chain rule

Step	What to do
1	Spot the inside function.
2	Differentiate the inside function mentally.
3	Check whether that derivative is present as a factor.
4	Adjust any constant multiplier.
5	Integrate and include + c.

Example 1: reverse chain rule with a power

Question: Integrate the expression shown below.

Working:

$$\int 6x(x^2 + 1)^4 dx$$

$$6x = 3(2x), \quad \frac{d}{dx}(x^2 + 1) = 2x$$

$$= 3 \int 2x(x^2 + 1)^4 dx$$

$$= \frac{3}{5}(x^2 + 1)^5 + c$$

Example 2: reverse chain rule with a logarithm

Question: Integrate the expression shown below.

Working:

$$\int \frac{6x}{3x^2 + 5} dx$$

$$\frac{d}{dx}(3x^2 + 5) = 6x$$

$$= \ln|3x^2 + 5| + c$$

Guided practice 1

Question	Space for working
1. Integrate $4x(x^2 + 3)^5$.	
2. Integrate $10x(5x^2 - 1)^3$.	
3. Integrate $3(3x + 2)^6$.	
4. Integrate $2x(x^2 + 7)^{-1}$.	
5. Integrate $12x(4x^2 + 1)^{-1}$.	
6. Explain how you can spot a reverse chain rule question.	

2. Integration by substitution

Substitution is a more formal version of the reverse chain rule. It changes the integral into a new variable, usually called u , so that it becomes easier to integrate.

Key idea

Choose u to be the awkward inside expression. Then replace every part of the integral so it is written in terms of u .

$$u = g(x), \quad \frac{du}{dx} = g'(x), \quad dx = \frac{du}{g'(x)}$$

Method: substitution

Step	What to do
1	Choose u to be the inside expression.
2	Differentiate u with respect to x .
3	Replace the x parts of the integral with u and du .
4	Integrate in terms of u .
5	Substitute back to x unless the question uses limits in u .

Example 3: substitution with a square root

Question: Integrate the expression shown below.

Working:

$$\int 2x\sqrt{x^2 + 4} \, dx$$

$$u = x^2 + 4, \quad \frac{du}{dx} = 2x$$

$$\int 2x\sqrt{x^2 + 4} \, dx = \int u^{\frac{1}{2}} \, du$$

$$= \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}(x^2 + 4)^{\frac{3}{2}} + c$$

Guided practice 2

Question	Space for working
1. Use substitution to integrate $2x(x^2 + 5)^4$.	
2. Use substitution to integrate $3x^2(x^3 + 1)^5$.	
3. Use substitution to integrate $2x\sqrt{x^2 + 9}$.	
4. Use substitution to integrate $x(4 - x^2)^5$.	
5. Use substitution to integrate e^{2x} .	
6. Explain why all x terms must be replaced before integrating in u.	

3. Integration by parts

Integration by parts is useful when the integrand is a product of two different types of function, such as x multiplied by an exponential or a logarithm.

Key formula

A common way to remember the formula is: integral of first times derivative of second equals first times second minus integral of second times derivative of first.

$$\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$$

Method: integration by parts

Step	What to do
1	Choose u as the part that becomes simpler when differentiated.
2	Choose dv over dx as the part that is easy to integrate.
3	Find du over dx and v.
4	Substitute into the formula.
5	Simplify and add + c where needed.

Example 4: integration by parts

Question: Integrate the expression shown below.

Working:

$$\int x e^x \, dx$$

$$u = x, \quad \frac{dv}{dx} = e^x, \quad \frac{du}{dx} = 1, \quad v = e^x$$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + c = e^x(x - 1) + c$$

Example 5: definite integration by parts

Question: Evaluate the definite integral shown below.

Working:

$$\int_0^1 xe^{2x} dx$$

$$u = x, \quad \frac{dv}{dx} = e^{2x}, \quad \frac{du}{dx} = 1, \quad v = \frac{1}{2}e^{2x}$$

$$= \left[\frac{x}{2} e^{2x} \right]_0^1 - \int_0^1 \frac{1}{2} e^{2x} dx$$

$$= \frac{e^2}{2} - \left[\frac{1}{4} e^{2x} \right]_0^1 = \frac{e^2 + 1}{4}$$

Guided practice 3

Question	Space for working
1. Integrate $x e^x$.	
2. Integrate $2x e^x$.	
3. Integrate $x \cos x$.	
4. Integrate $x \sin x$.	
5. Evaluate the integral of $x e^x$ from 0 to 1.	
6. Explain how to choose u in integration by parts.	

4. Integrating partial fractions

Some algebraic fractions can be integrated by splitting them into partial fractions first. This often turns one difficult fraction into two simple logarithmic integrals.

Key idea

Factorise the denominator first. Then split the fraction into simpler fractions and integrate each one separately.

Method: partial fractions before integrating

Step	What to do
1	Check that the denominator is factorised.
2	Write the expression as a sum of partial fractions.
3	Find the unknown constants.
4	Integrate each separate fraction.
5	Use logarithms for terms of the form constant over a linear factor.

Example 6: integrating partial fractions

Question: Integrate the expression shown below.

Working:

$$\int \frac{5x + 1}{(x + 1)(x + 2)} dx$$

$$\frac{5x + 1}{(x + 1)(x + 2)} = \frac{A}{x + 1} + \frac{B}{x + 2}$$

$$5x + 1 = A(x + 2) + B(x + 1)$$

$$A = -4, \quad B = 9$$

$$\int \frac{5x + 1}{(x + 1)(x + 2)} dx = -4\ln|x + 1| + 9\ln|x + 2| + c$$

Guided practice 4

Question	Space for working
1. Decompose and integrate $(3x + 5)((x + 1)(x + 4))^{-1}$.	
2. Decompose and integrate $(2x + 7)((x + 2)(x + 3))^{-1}$.	
3. Decompose and integrate $(5x - 1)((x - 1)(x + 2))^{-1}$.	
4. Integrate $4(x + 3)^{-1}$.	
5. Integrate $6(2x - 1)^{-1}$.	
6. Explain why logarithms appear when integrating simple partial fractions.	

5. Trigonometric integration

Some trigonometric integrals can be handled directly from known results. Others need a constant adjustment when the angle is a multiple of x .

Key facts

These are common trigonometric integrals for A-Level Maths.

$$\int \sin x \, dx = -\cos x + c, \quad \int \cos x \, dx = \sin x + c, \quad \int \sec^2 x \, dx = \tan x + c$$

Method: trigonometric integration

Step	What to do
1	Identify the basic trig function.
2	Check whether the angle is x or a multiple of x .
3	If the angle is kx , divide by k when reversing the chain rule.
4	Watch signs carefully, especially for sine and cosine.
5	Add $+ c$ for indefinite integrals.

Example 7: integrating a trig function

Question: Integrate the expression shown below.

Working:

$$\int \sin(3x) \, dx$$

$$\int \sin(3x) \, dx = -\frac{1}{3}\cos(3x) + c$$

Guided practice 5

Question	Space for working
1. Integrate $\cos x$.	
2. Integrate $\sin x$.	
3. Integrate $\cos(2x)$.	
4. Integrate $\sin(5x)$.	
5. Integrate $\sec^2(3x)$.	
6. Explain why integrating $\sin(3x)$ gives a factor of one third.	

6. Mixed exam-style questions

Question	Space for working
1. [3 marks] Integrate $8x(x^2 + 4)^3$.	
2. [3 marks] Integrate $5(5x - 2)^6$.	
3. [4 marks] Use substitution to integrate $2x\sqrt{x^2 + 1}$.	

Question	Space for working
4. [4 marks] Integrate $x e^x$.	
5. [4 marks] Integrate $x \cos x$.	
6. [4 marks] Integrate $(4x + 1)((x + 1)(x + 3))^{-1}$.	
7. [3 marks] Integrate $\sin(4x)$.	
8. [4 marks] Evaluate the integral of $x e^{2x}$ from 0 to 1.	
9. [5 marks] Integrate $(7x + 1)((x - 1)(x + 2))^{-1}$.	
10. [5 marks] A curve has gradient $6x(x^2 + 2)^4$ and passes through (0, 3). Find its equation.	

7. Challenge questions

Question	Space for working
C1. Use substitution to integrate $x(2x^2 + 5)^{-1}$.	
C2. Use integration by parts to integrate $x^2 e^x$. You may need to apply the method twice.	
C3. Find the exact value of the integral of $x \cos(2x)$ from 0 to π .	
C4. Integrate $(3x^2 + 2)(x^3 + 2x - 1)^5$.	
C5. Find the exact value of the integral of $\ln x$ from 1 to e using integration by parts.	

8. Common mistakes

Mistake	How to fix it
Trying to use the reverse chain rule when the inside derivative is missing.	Differentiate the inside expression first and check whether that derivative is present.
Forgetting to change dx during substitution.	Use du over dx to replace the whole x part of the integral.
Choosing u badly in integration by parts.	Choose u so it becomes simpler when differentiated.
Forgetting the minus sign in integration by parts.	Write the formula before substituting values.
Splitting algebraic fractions without factorising the denominator.	Factorise the denominator before setting up partial fractions.
Missing the constant adjustment in trig integrals.	If the angle is kx , divide by k when integrating.

9. Fully worked answers

Guided practice 1

- Two fifths of $(x^2 + 3)^6$ plus c .
- $(5x^2 - 1)^4$ over 4 plus c .
- $(3x + 2)^7$ over 7 plus c .
- $\ln|x^2 + 7|$ plus c .
- Three halves $\ln|4x^2 + 1|$ plus c .
- The inside function and a multiple of its derivative appear together.

Guided practice 2

1. Let $u = x^2 + 5$. The answer is one third of $(x^2 + 5)^3$ plus c .
2. Let $u = x^3 + 1$. The answer is $(x^3 + 1)^6$ over 6 plus c .
3. Let $u = x^2 + 9$. The answer is two thirds of $(x^2 + 9)$ to the power three halves plus c .
4. Let $u = 4 - x^2$. The answer is negative one twelfth of $(4 - x^2)^6$ plus c .
5. The answer is one half e^{2x} plus c .
6. An integral in u must contain only u and du . Any leftover x means the substitution is incomplete.

Guided practice 3

1. $e^x(x - 1)$ plus c .
2. $2e^x(x - 1)$ plus c .
3. $x \sin x + \cos x$ plus c .
4. $-x \cos x + \sin x$ plus c .
5. The exact value is 1.
6. Choose u as the part that gets simpler when differentiated, such as x or $\ln x$.

Guided practice 4

1. $(3x + 5)((x + 1)(x + 4))^{-1} =$ two thirds over $x + 1$ plus seven thirds over $x + 4$. Integrate to get two thirds $\ln|x + 1|$ plus seven thirds $\ln|x + 4|$ plus c .
2. The answer is $3 \ln|x + 2| - \ln|x + 3|$ plus c .
3. The answer is $2 \ln|x - 1| + 3 \ln|x + 2|$ plus c .
4. $4 \ln|x + 3|$ plus c .
5. $3 \ln|2x - 1|$ plus c .
6. Integrating a constant over a linear expression gives a logarithm.

Guided practice 5

1. $\sin x$ plus c .
2. $-\cos x$ plus c .
3. one half $\sin(2x)$ plus c .
4. negative one fifth $\cos(5x)$ plus c .
5. one third $\tan(3x)$ plus c .
6. The derivative of $3x$ is 3, so the reverse chain rule needs a factor of one third.

Mixed exam-style questions

1. $(x^2 + 4)^4$ plus c .
2. $(5x - 2)^7$ over 7 plus c .
3. Two thirds of $(x^2 + 1)$ to the power three halves plus c .
4. $e^x(x - 1)$ plus c .
5. $x \sin x + \cos x$ plus c .
6. Decompose into negative one half over $x + 1$ plus nine halves over $x + 3$. The integral is negative one half $\ln|x + 1|$ plus nine halves $\ln|x + 3|$ plus c .
7. negative one quarter $\cos(4x)$ plus c .
8. The exact value is $(e^2 + 1)$ over 4.
9. Decompose into 8 thirds over $x - 1$ plus 13 thirds over $x + 2$. Integrate to get 8 thirds $\ln|x - 1|$ plus 13 thirds $\ln|x + 2|$ plus c .
10. The equation is three fifths of $(x^2 + 2)^5$ plus c . Using $(0, 3)$, $c = 3 - 96$ fifths, so $c =$ negative 81 fifths.

Challenge questions

- C1. one quarter $\ln|2x^2 + 5|$ plus c .
- C2. $e^x(x^2 - 2x + 2)$ plus c .

C3. The exact value is zero.

C4. one half of $(x^3 + 2x - 1)^6$ plus c.

C5. Treat $\ln x$ as $\ln x$ times 1. The exact value is 1.

10. Revision checklist

Skill	Tick
I can spot reverse chain rule integrals.	☐
I can use logarithms when integrating f prime over f.	☐
I can use substitution and replace all x terms.	☐
I can use integration by parts correctly.	☐
I can integrate algebraic fractions after partial fractions.	☐
I can integrate simple trigonometric functions.	☐
I can handle constant multipliers from inside functions.	☐
I can show clear working in exam-style integration questions.	☐